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LETTER TO THE EDITOR

Cubic against isotropic critical behaviour for long-range interactions

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Abstract. The relevance of cubic perturbations in d-dimensional n-vector models with long-range interactions $V(x) \propto x^{-d-\sigma}$, $\sigma > 0$ is shown to depend on d, n and σ in the non-classical long-range region. These long-range interactions favour isotropic behaviour, but it is shown that there are physical systems with $n = 3$ or even $n = 2$ in two and one space dimensions which display cubic behaviour for appropriate values of σ. These results were derived by using the $1/n$ correction to the large n limit in order to obtain an upper limit for the isotropic region.

The stability of the isotropic fixed point behaviour against cubic perturbations has been studied by the renormalization group (RG) (Aharony 1973, Grover *et al* 1972) and by modified ϵ and $1/n$ expansions (Ketley and Wallace 1973) for short-range (SR) interactions. Due to the asymptotic character of the ϵ expansion one can only estimate that $2 < \bar{n}(d) \leqslant 4$ for $3 \lesssim d\,(\leqslant 4)$ (a numerical RG study gives $\bar{n}(3) \gtrsim 3$), where $\bar{n}(d)$ is the number of field components which separates isotropic behaviour found for small $n < \bar{n}$ from the cubic behaviour at $n > \bar{n}$. The $O(1/n)$ result of the $1/n$ expansion, $\bar{n}(3) = 5{\cdot}3$, represents rather an upper limit than a numerically satisfying value for the true $\bar{n}$.

In this letter we give some results to $O(1/n)$ and to $O(\epsilon)$ concerning the dependence of the borderline $\bar{n}(d, \sigma)$ on the LR parameter σ defined by the potential $V(x) \propto x^{-d-\sigma}$, $0 < \sigma < 2$. The hamiltonian considered is $\beta H = \beta H_0 + \beta H_i$ with

$$\begin{aligned}\beta H_0 &= \tfrac{1}{2}\int d^dx \sum_{\alpha=1}^{n} [r_0 S_\alpha^2(x) + (\nabla^{\sigma/2} S_\alpha(x))^2] \\ \beta H_i &= \int d^dx \left[\frac{u_0}{4!}\left(\sum_{\alpha=1}^{n} S_\alpha^2(x)\right)^2 - \frac{\Delta}{4}\sum_{\alpha=1}^{n} S_\alpha^4(x)\right]\end{aligned} \tag{1}$$

where $r_0 \propto T - T_{c0}$ and $\Delta \ll u_0$ is assumed. The RG study (Aharony 1973) shows that in the cubic region this assumption can only be satisfied by positive Δ of $O(1/n)$ but not for $0 > \Gamma = O(1)$, since $u_0 = O(1/n)$. The following results for LR scaling are valid in the 'restricted nonclassical LR region' (Sak 1973), where $\frac{1}{2}d < \sigma < d$ and $\sigma < 2 - \eta_{sr}$ for $2 < d \leqslant 4$. Here, we conjecture that the restriction on the LR region via $\eta_{lr} > \eta_{sr}$ appears in the $1/n$ expansion too. It is remarkable that this condition coincides in $O(1/n)$ with $\sigma < 2$ for $d \to 2^+$, since $\lim_{d\to 2^+} \eta_{sr} = 0 + O(n^{-2})$.

Applying the graphical method of Ketley and Wallace (1973), we study the competition between isotropic and cubic scaling by considering the connected four-spin correlation function $u_{ijkl} = \int_{xyz}\langle s_{ix}s_{jy}s_{kz}s_{l0}\rangle^c/(\int_x\langle s_{ix}s_{i0}\rangle)^4$ with its decomposition $u_{ijkl} = u_1(\delta_{ij}\delta_{kl}+\text{perm})+u_2\delta_{ijkl}$, where $\delta_{ijkl} = 1$ if $i = j = k = l$ and zero otherwise. The isotropic fixed point is stable if $\alpha_2 > \alpha_1$, and unstable if $\alpha_2 < \alpha_1$, where α_2, α_1 are the critical exponents of the 'cubic function' $u_2 \propto \Delta r^{\alpha_2}$ and the 'isotropic one' $u_1 \propto u_0 r^{\alpha_1}$, respectively, both taken to lowest order in Δ. Here, r is the inverse susceptibility. As usual, we define $\epsilon = 2\sigma-d$. By using $\eta = 2-\sigma$, it follows from scaling laws (Wilson and Kogut 1972) that $\alpha_1 = \epsilon/\sigma$ to $O(1/n)$ for all ϵ and to $O(\epsilon)$ for all n.

The $O(1/n)$ term of u_2 is a Δ vertex which is renormalized by a stream of bubbles. Our result takes the form

$$u_2 = 6\Delta\left(1+\frac{24}{n\sigma}\frac{\Gamma(d-\sigma)\Gamma^2(\frac{1}{2}\sigma)}{\Gamma(\frac{1}{2}d)\Gamma^2(\frac{1}{2}d-\frac{1}{2}\sigma)\Gamma(\sigma-\frac{1}{2}d)}\ln r\right) \tag{2}$$

so that the critical exponent α_2 reads

$$\alpha_2 = \frac{24}{n\sigma}\frac{\Gamma(d-\sigma)\Gamma^2(\frac{1}{2}\sigma)}{\Gamma(\frac{1}{2}d)\Gamma^2(\frac{1}{2}d-\frac{1}{2}\sigma)\Gamma(\sigma-\frac{1}{2}d)}. \tag{3}$$

The critical number $\bar{n}(d,\sigma)$ of field components is determined by $\alpha_2(\bar{n})-\alpha_1(\bar{n}) = 0$, which yields

$$\bar{n}(d,\sigma) = \frac{24}{2\sigma-d}\frac{\Gamma(d-\sigma)\Gamma^2(\frac{1}{2}\sigma)}{\Gamma(\frac{1}{2}d)\Gamma^2(\frac{1}{2}d-\frac{1}{2}\sigma)\Gamma(\sigma-\frac{1}{2}d)}. \tag{4}$$

This result is illustrated in the n–ϵ diagram of figure 1 for the discrete space dimensions of interest. For $d = 3$ one can observe in figure 1 that the restriction of the LR region causes a continuous changeover from the LR to the SR value of $\bar{n}$ at $2-\sigma = \eta_{sr}$. For different dimensions d the same values of ϵ rather than σ should be compared. Figure 1 shows that for fixed d the limiting value $\bar{n}$ increases with decreasing σ, ie LR interactions favour isotropic behaviour. However, our most important conclusion is that the $O(1/n)$ result (4) predicts the existence of σ regions, where one- and two-dimensional physical systems with three or even two field components display cubic behaviour. In the case

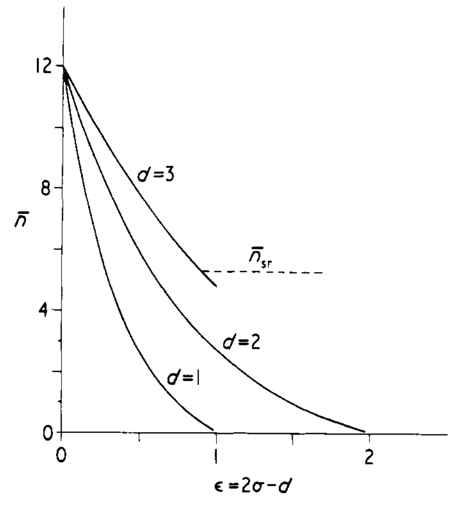

Figure 1. Critical number $\bar{n}$ of field components (separating cubic behaviour for $n > \bar{n}$ from isotropic behaviour for $n < \bar{n}$) as function of $\epsilon = 2\sigma-d$ for $d = 3$, $d = 2$ and $d = 1$.

$d = 2$ one finds for example cubic behaviour for $n = 3$ if $\epsilon \gtrsim 0{\cdot}9$ or $\sigma \gtrsim 1{\cdot}45$ and for $n = 2$ if $\epsilon \gtrsim 1{\cdot}2$ or $\sigma \gtrsim 1{\cdot}6$, respectively. One can expect the cubic regions to be even larger, because the $1/n$ correction to the large n limit overestimates the isotropic contributions from small n. This property of the $1/n$ expansion is well understood and we demonstrate it for small ϵ, where the ϵ expansion to $O(\epsilon)$ yields the exact $\bar{n}$ for $\epsilon \to 0$.

In the ϵ expansion we obtain to $O(\epsilon)$

$$\alpha_2 - \alpha_1 = \frac{4-n}{n+8}\frac{\epsilon}{\sigma} + O(\epsilon^2), \tag{5}$$

where $u_0(\epsilon) = K_{2\sigma}^{-1}(n+8)^{-1}\epsilon/4 + O(\epsilon^2)$ with $K_{2\sigma}^{-1} = 2^{2\sigma-1}\pi^{\sigma}\Gamma(\sigma)$ has been used. Thus, the $O(\epsilon)$ result gives $\bar{n} = 4$ independent of σ, while the $O(1/n)$ result shows a significant σ dependence for all ϵ and gives $\bar{n} = 12$ for $\epsilon \to 0$. Again as in the SR case the competing $O(\epsilon^2)$ and $O(\epsilon^3)$ contributions cannot be expected to yield a reasonable value $\bar{n}$ for $\epsilon \gtrsim 1$. We note that the results (5) and (3) with $\alpha_1 = \epsilon/\sigma$ are consistent for small ϵ and large n.

Finally, we turn to formula (3) and remark that $\lim_{\sigma\to 2}\alpha_2(\sigma) = \alpha_2^{\mathrm{sr}}$. Therefore equation (3) is valid in the whole nonclassical LR region $\frac{1}{2}d < \sigma < \min(2, d)$. This property is also exhibited by the related critical exponent ψ of the transverse susceptibility which behaves as $\chi_T \propto M^{-\psi}$ below T_c. An independent $1/n$ expansion to $O(1/n)$ yields

$$\psi = 2 + \frac{48}{n}\frac{\Gamma(d-\sigma)\Gamma^2(\frac{1}{2}\sigma)(d-\sigma)^{-1}}{\Gamma(\frac{1}{2}d)\Gamma^2(\frac{1}{2}d-\frac{1}{2}\sigma)\Gamma(\sigma-\frac{1}{2}d)}. \tag{6}$$

Thus, our results for ψ, α_2 and α_1 satisfy Wallace's scaling relation $\psi = (\delta-1)(1+\alpha_2-\alpha_1)$, in which the 'irrelevant discontinuities' of δ and α_1 at $\sigma = 2$ compensate. In the crossover region from cubic to isotropic behaviour at $\bar{n}$ one finds by $1/n$ expansion to $O(1/n)$ of the SR and LR equation of state (Wallace 1973) that $\phi = \beta\omega = \gamma(\alpha_1-\alpha_2) = \gamma - \beta\psi$ for $0 \leqslant \Delta = O(1/n)$, where ϕ and ω are defined by $\Delta \propto \tau^{\phi}$ and $M \propto \Delta^{1/\omega}$. The relation between M and Δ also describes the discontinuous change of the order parameter at T_c in the cubic region $0 < \Delta = O(1/n)$, $n > \bar{n}$. Since ϕ, ω and $\alpha_1 - \alpha_2$ approach zero at the same value $\bar{n}$, Wallace's first-order transition vanishes when the cubic perturbation becomes irrelevant, as is to be expected.

Further results including ϵ expansions to higher orders will be published elsewhere.

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